TRANSMISSION LINES AND MATCHING

for
High-Frequency Circuit Design Elective

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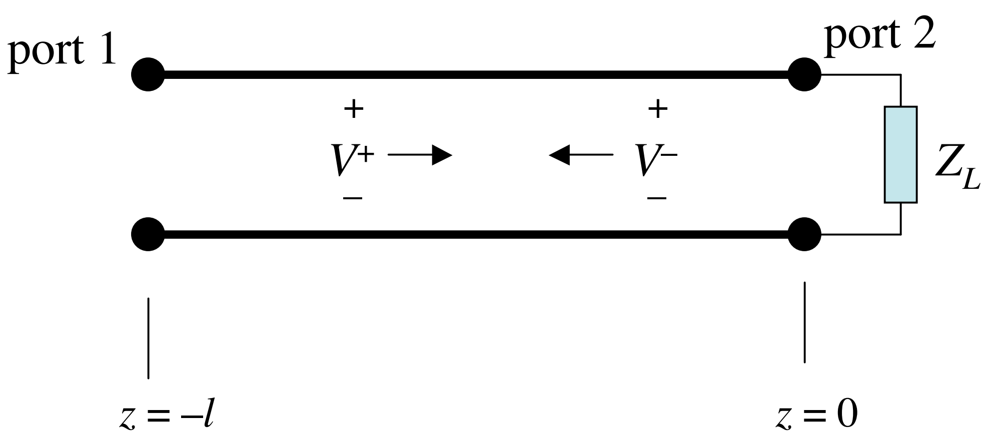
Contents

Transmission Lines and Matching

IMPORTANT: Our discussion assumes that the frequency is so high that the transmission line is of comparable dimension to the signal wavelength. Hence, ***lumped circuit models are invalid.***

Basic model

Two waves travelling in opposite directions.



z = distance from the terminating resistance

Key idea: voltage is varying with z; there are two travelling voltages $V^+(z)$ and $V^-(z)$

Suppose the transmission line has an inductance per unit length of L and a capacitance per unit length of C.

$$\text{Inductance for } dz = L\,dz \quad \text{and} \quad \text{Capacitance for } dz = C\,dz$$

Suppose a voltage V and a current I enters from the left end (port 1) and travel through the transmission line.

At port 1: $\quad v_1 = V \quad \text{and} \quad i_1 = I$

At port 2: $\quad v_2 = V + \dfrac{\partial V(t,z)}{\partial z} dz$

The voltage difference between the two end ports is

$$v_2 - v_1 = \frac{\partial V(t,z)}{\partial z} dz = -L dz \frac{\partial I(t,z)}{\partial t}$$

Similarly, $$i_2 - i_1 = \frac{\partial I(t,z)}{\partial z} dz = -C dz \frac{\partial V(t,z)}{\partial t}$$

Hence, we obtain

$$\frac{\partial V(t,z)}{\partial z} = -L\frac{\partial I(t,z)}{\partial t} \qquad (*)$$

$$\frac{\partial I(t,z)}{\partial z} = -C\frac{\partial V(t,z)}{\partial t} \qquad (**)$$

Differentiating (*) with respect to z and (**) with respect to t, we get

$$\frac{\partial^2 V(t,z)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 V(t,z)}{\partial t^2}$$

$$\frac{\partial^2 I(t,z)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 V(t,z)}{\partial t^2}$$

The famous
Telegrapher's equations

where v is velocity of the wave given by

$$v = \frac{1}{\sqrt{LC}}$$

Solutions of the Telegrapher's Equations

The Telegrapher's equations are actually standard wave equations. The solution (from physics) is

$$V(t,z) = F_1\left(t - \frac{z}{v}\right) + F_2\left(t + \frac{z}{v}\right)$$

$$V(t,z) = \frac{1}{Z_o}\left[F_1\left(t - \frac{z}{v}\right) + F_2\left(t + \frac{z}{v}\right)\right]$$

where $F_1(.)$ and $F_2(.)$ are the forward and backward travelling wave functions.

For sine waves, $F_1(.)$ and $F_2(.)$ are exponential functions $e^{j(\omega t \pm \beta z)}$, where $\beta = \omega/v$.

Also, Z_o is the *characteristic impedance* of the transmission line

$$Z_o = \sqrt{\frac{L}{C}} = Lv = \frac{1}{Cv}$$

Transmission Line Equations

Suppose at the load side (i.e., port 2), the travelling voltages are $V^+(0)$ and $V^-(0)$. So, the voltage at the load is simply the sum of $V^+(0)$ and $V^-(0)$. But the current should be the difference of $I^+(0)$ and $I^-(0)$ because they flow in opposite directions in the transmission line.

$$V_L = V^+(0) + V^-(0) \qquad (\#)$$

$$I_L = I^+(0) - I^-(0) = \frac{V^+(0) - V^-(0)}{Z_o} \qquad (\#\#)$$

But the current must be consistent with Ohm's law at the load. Hence,

$$I_L = \frac{V^+(0) - V^-(0)}{Z_o} = \frac{V_L}{Z_L} \qquad (\#\#\#)$$

So, (#) and (###) give

$$\boxed{\frac{V^-(0)}{V^+(0)} = \frac{Z_L - Z_o}{Z_L + Z_o} \quad \text{defined as reflection coefficient } \Gamma}$$

The voltage and current at any position z are

$$V(z) = V^+(0)e^{-j\beta z} + V^-(0)e^{+j\beta z}$$

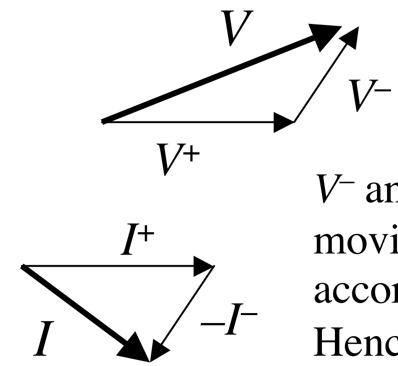
$$I(z) = \frac{V^+(0)e^{-j\beta z} - V^-(0)e^{+j\beta z}}{Z_o}$$

Thus, at position z, the input impedance is

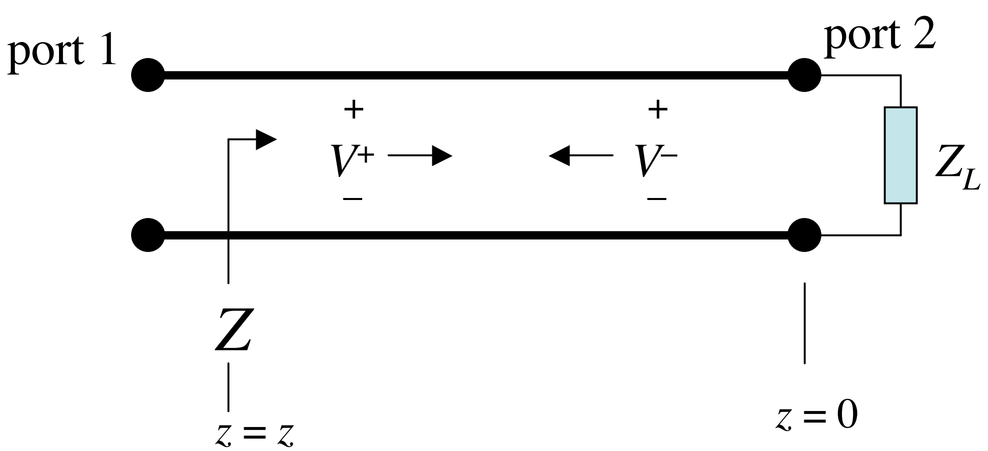
$$Z = Z_o \frac{V^+(0)e^{-j\beta z} + V^-(0)e^{+j\beta z}}{V^+(0)e^{-j\beta z} - V^-(0)e^{+j\beta z}}$$

$$= Z_o \frac{e^{-j\beta z} + \Gamma e^{+j\beta z}}{e^{-j\beta z} - \Gamma e^{+j\beta z}} = Z_o \frac{1 + \Gamma e^{2j\beta z}}{1 - \Gamma e^{2j\beta z}}$$

Phasor view

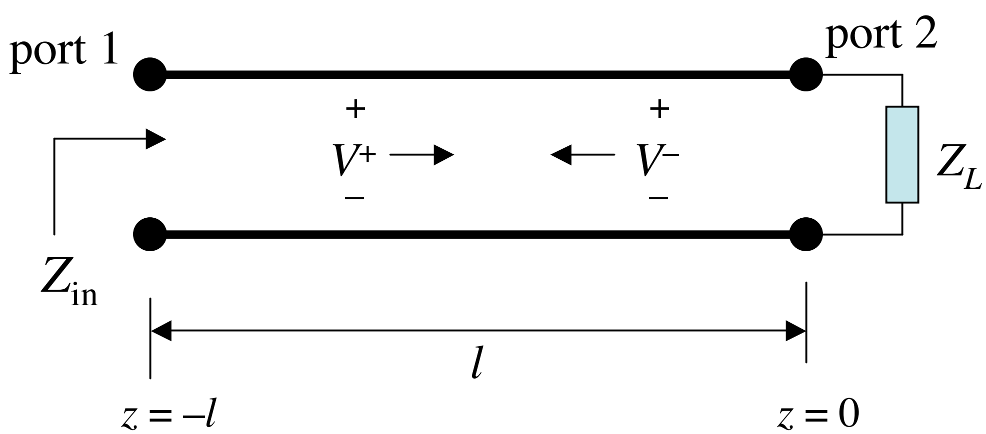


V^- and I^- are moving according to Γ. Hence, Z is changing too.



At the left end of the transmission line (port 1), the input impedance is

$$Z_{\text{in}} = Z_o \frac{1+\Gamma e^{-2j\beta l}}{1-\Gamma e^{-2j\beta l}} = Z_o \frac{Z_L + jZ_o \tan\beta l}{Z_o + jZ_L \tan\beta l}$$



Impedance for any length l can be found from this equation.

Special Cases

If the load is matched, i.e., $Z_L = Z_o$, then

- $\Gamma = 0$
- $Z_{\text{in}} = Z_L$ for all l.

If the transmission line is a quarter wavelength long ($l = \lambda/4$), $Z_{\text{in}} = Z_o^2/Z_L$.

If the transmission line is a half wavelength long ($l = \lambda/2$), $Z_{\text{in}} = Z_L$.

If the load is open-circuit, i.e., $Z_L = \infty$, then $\Gamma = 1$.

If the load is short-circuit, i.e., $Z_L = 0$, then $\Gamma = -1$.

Smith Chart

The general impedance equation is $$\frac{Z_{\text{in}}}{Z_o} = \frac{1+\Gamma e^{-2j\beta l}}{1-\Gamma e^{-2j\beta l}}$$

$\Gamma e^{-2j\beta l} = \Gamma(l)$ reflection coeff. at the distance l from load

In general, Z_{in}/Z_o is a complex number ζ. So, we may assume that

$$\frac{Z_{\text{in}}}{Z_o} = \zeta = r + jx$$

Also, in general, Γ is a complex number whose magnitude is between 0 and 1. (E.g., 0 for matched load, 1 for open-circuit, –1 for short-circuit, etc.)

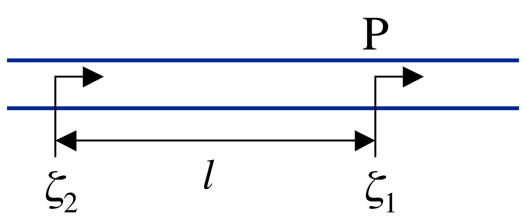
We may rewrite the equation as

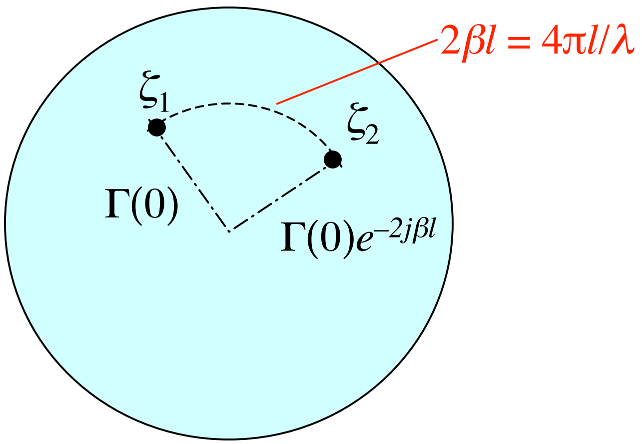
$$\zeta = \frac{1+\Gamma(l)}{1-\Gamma(l)} \qquad \text{or} \qquad \Gamma(l) = \frac{\zeta - 1}{\zeta + 1} = \frac{(r-1)+jx}{(r+1)+jx}$$

The Smith Chart as a Calculation Tool

Basically the Smith chart is a polar plot of the reflection coefficient. Usually we put the *normalized impedance* ζ on top of this polar plot.

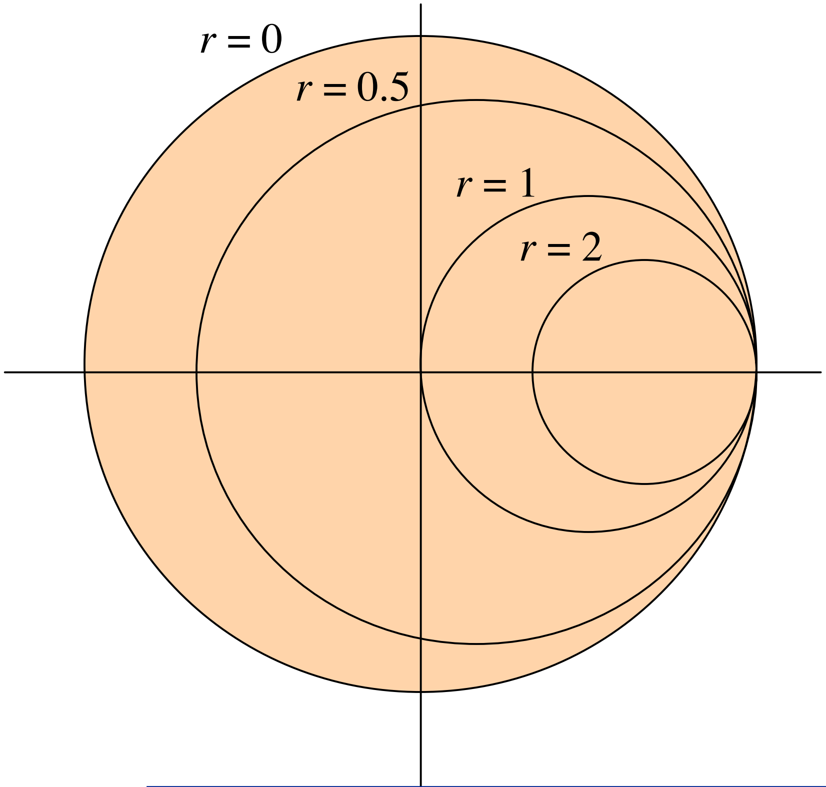
If we know the impedance at a certain point P along the transmission line, then the impedance at a distance l from this point can be read off from the Smith chart by transforming it appropriately according to the reflection coefficient Γ.

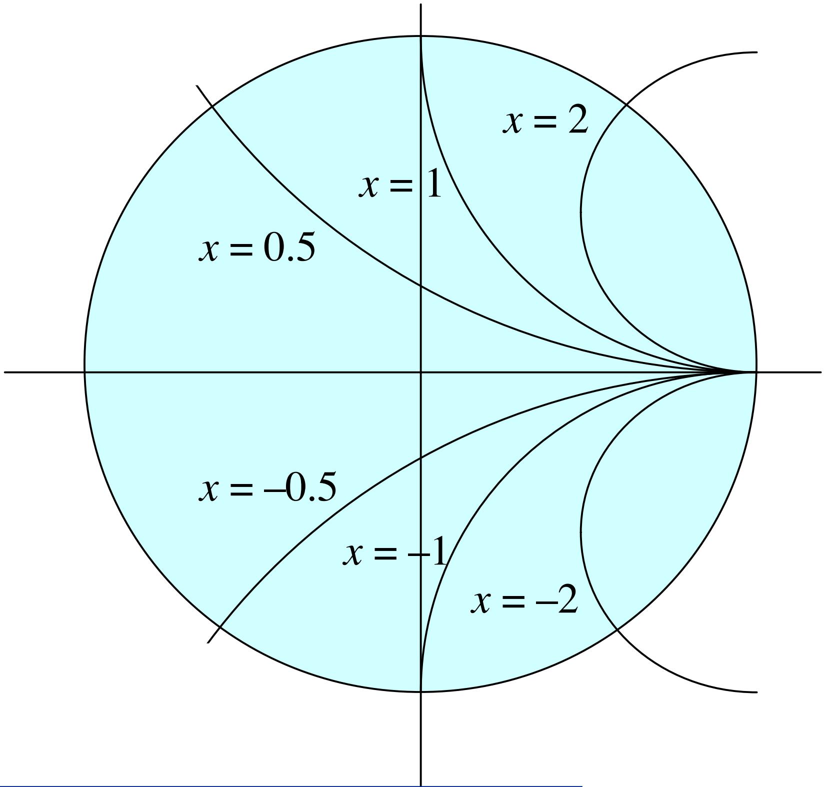




So, we may imagine that the magnitude $|\Gamma|$ is the radius and the argument of Γ is the rotating angle $2\beta l$.

The solution loci of the impedance equation can be mapped to

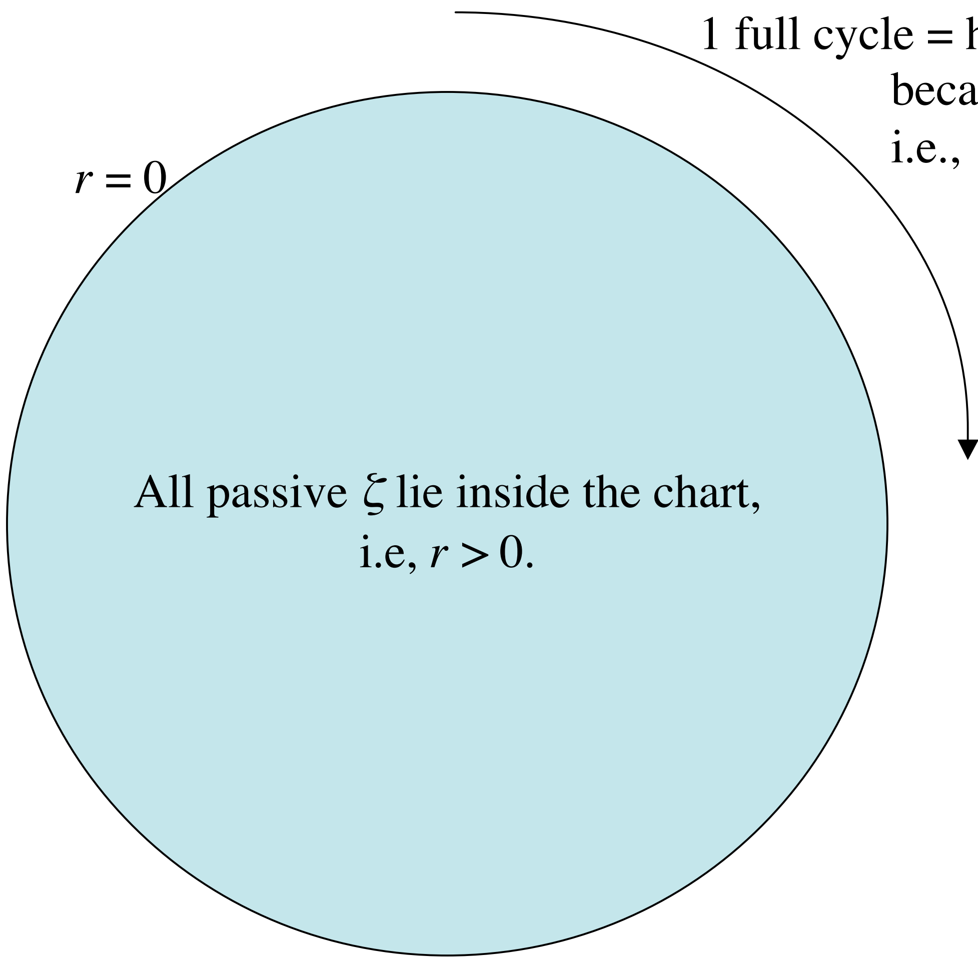




Real part r : circles centered at $\left(\frac{r}{1-r}, 0\right)$, with radius $\frac{1}{1-r}$

Imaginary part x : circles centered at $\left(1, \frac{1}{x}\right)$, with radius $\frac{1}{x}$

Using the Smith Chart



Usages:

- For lossless transmission lines, $|\Gamma|$ remains unchanged. Hence, any impedance point on the chart simply rotates in clockwise (generator) direction as the length increases.
- But for lossy lines, $|\Gamma|$ also decreases. Thus, the impedance point moves closer to the center as well as rotates, as the length increases.
- Also, turning it 180^o will make it an admittance chart! Easy!

Example

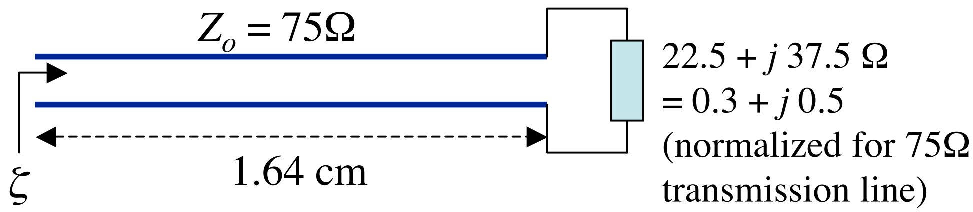
Suppose an impedance $22.5 + j\,37.5$ Ω is terminating a transmission line of $Z_o = 75$ Ω.

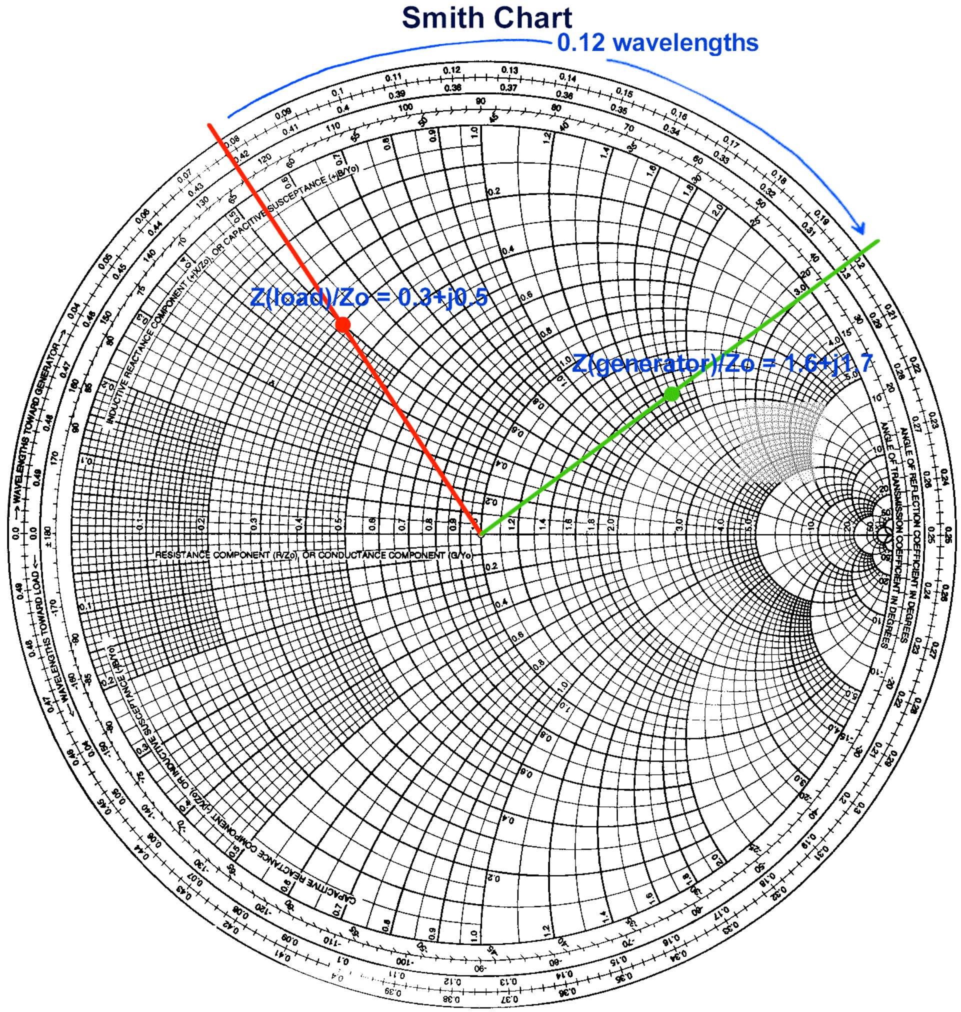
The line is 1.64 cm long and the frequency is 146 MHz.

Suppose the velocity factor is 2/3, i.e., the wave travels at 2/3 of the speed of light which is $2\text{x}10^8$ m/s or 20 cm/ns.

Thus, $\lambda = v\,/\,f = 1.37$ m, and the line length is 0.12λ.

On the chart, we put $0.3 + j\,0.5$ first, and then rotate it through 0.12λ. The transformed impedance is $1.6 + j\,1.7$, i.e., $120 + j\,127.5$ Ω for a 75Ω line.





Summary of Important Properties of Transmission Lines

Characteristic impedance, Z_o = ratio of the voltage to current of the forward travelling wave, assuming no backward wave. It is a *real but lossless* impedance!
$Z_o = \sqrt{L/C}$, where L is inductance per unit length and C is capacitance per unit length.

Wave velocity has two types. *Phase velocity* and *group velocity*. The phase velocity is the speed of a point of a chosen phase moving along the line. So, phase velocity is how fast a wave moves. If we find a point where $V^+ = 0$ and track this point, we will see that it moves at the phase velocity. Thus, phase velocity is ω/β or $f\lambda$. Moreover, group velocity is defined as $d\omega/d\beta$, which describes how fast energy travels along the line. For coax cables and parallel lines, phase velocity is same as group velocity, and is given by $1/\sqrt{LC}$.

Return loss is defined as $-20 \log_{10} |\Gamma|$ dB, and it is how much power is reflected. For instance, a return loss of 3 dB means that half the power is reflected, and a return loss of 20 dB means that 1% of the power is reflected.

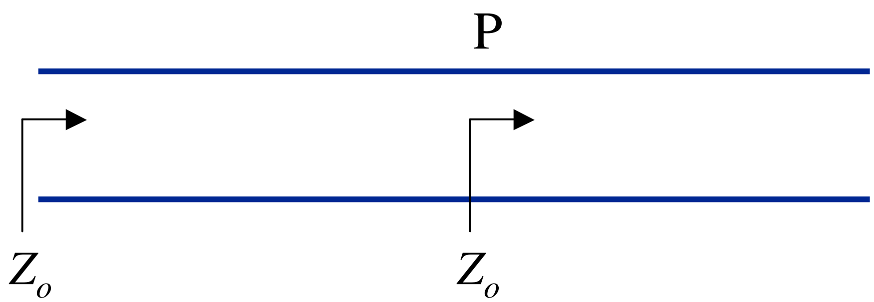
Voltage standing wave ratio (VSWR) is the ratio of the maximum voltage amplitude to the minimum voltage amplitude which is at $\lambda/4$ from the maximum point. VSWR is given by

$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

And it measures the extent of mismatch. If 1% of incident power is reflected by load, $|\Gamma|^2 = 0.01$ and $|\Gamma| = 0.1$, then VSWR ≈ 1.2. Its value is 0 for matched load.

Matching Condition

Suppose we have an imaginary joint at P in a very long transmission line.

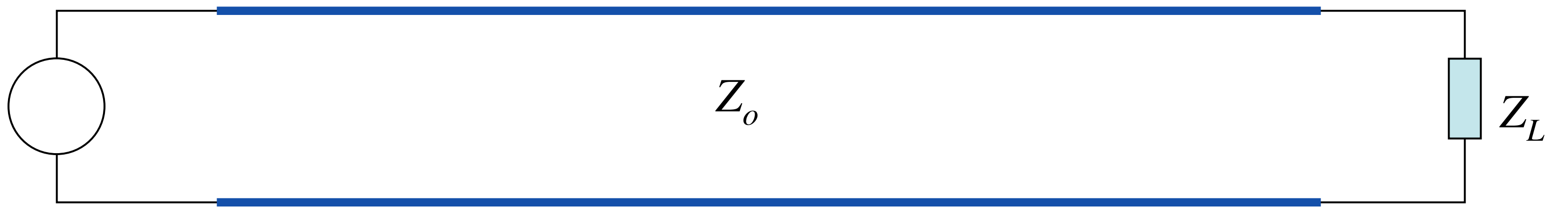


The wave goes through the joint without reflection because there is actually no joint (just imagined).

Now, let us terminate a resistance of value Z_o at the same position of this imaginary joint. Obviously, the wave will go through without reflection too.

This is called a **matched load.** So, we cannot tell if anything has happened because nothing gets reflected.

Why do we want to match a transmission line?

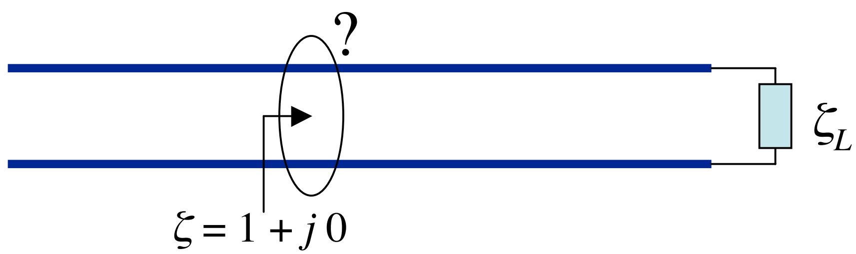


1. We can make sure that all power is delivered to the load circuit.

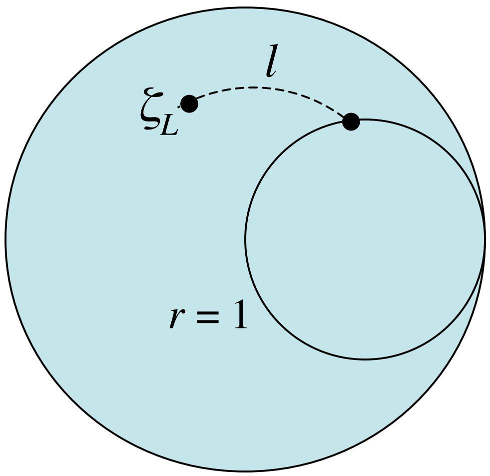
2. The generator (signal source) will not resonate with the line. Usually, the generator (signal source) is designed to drive a 50Ω or 75Ω load, which is the characteristic impedance of common transmission lines. If the load is matched with the line, the generator will see no reactive part and the length of the line will not affect anything.

Simplest Matching Strategy

Basically we want the impedance to match the characteristic impedance of the transmission line. In other words, we want at some point on the line, the normalized impedance $\zeta = 1 + j\,0$.

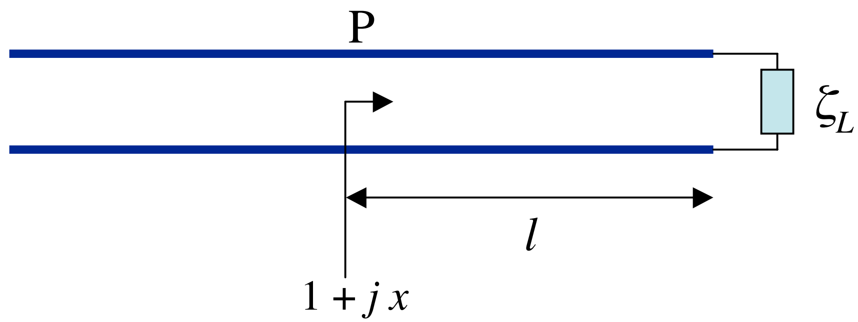


Look at the Smith chart! We can always find the circle corresponding to $r = 1$ which has the desired real part for matching.

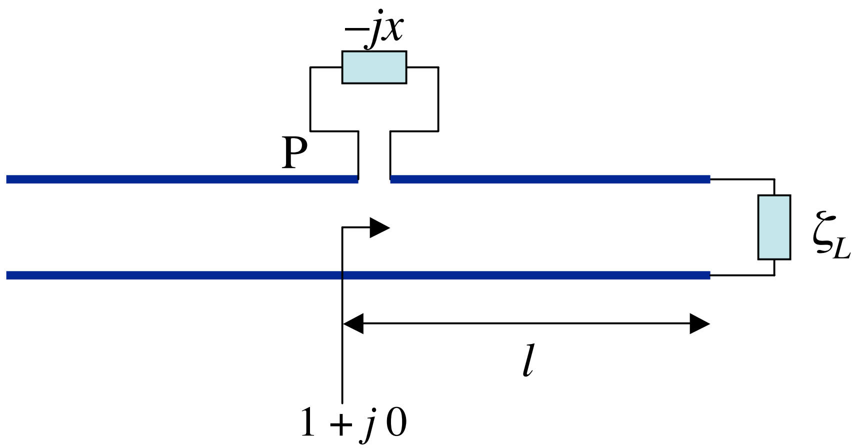


So, we can always find a point on the line where the real part r is 1. Let's say this point is l from the load.

Remember, l can be conveniently measured on the Smith chart (actually marked on the chart as number of wavelength).



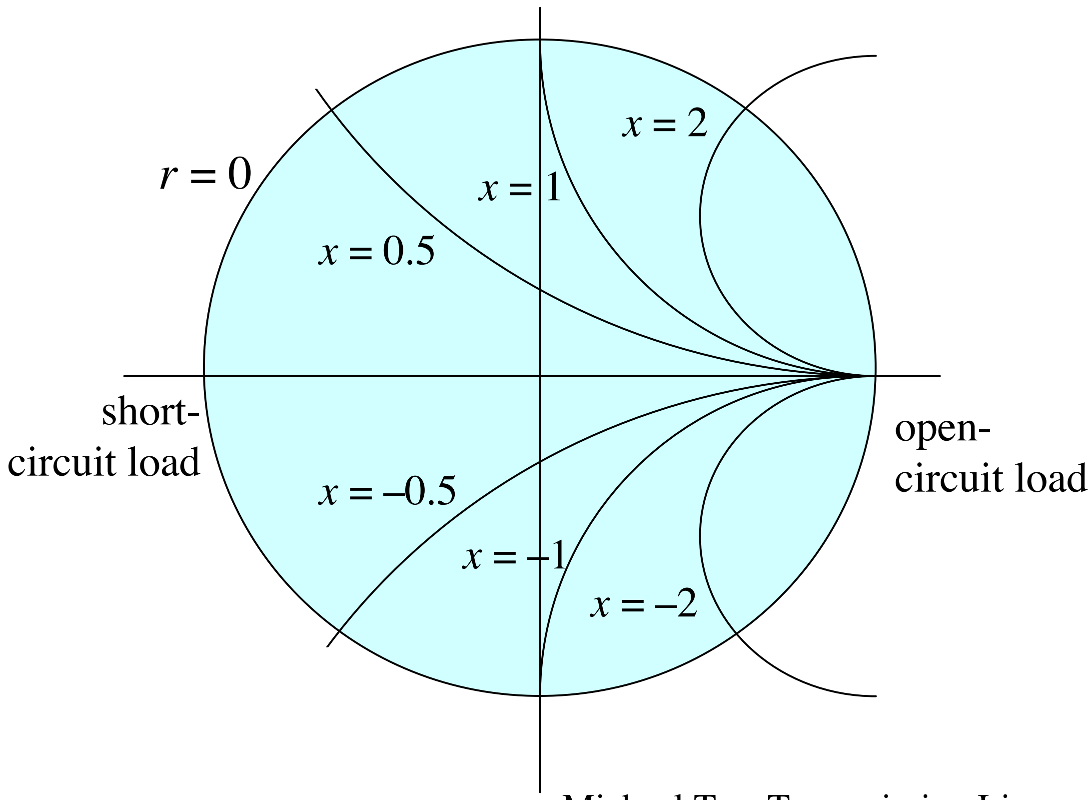
So, what remains to be done now is to cancel out the impedance jx by connecting at point P a reactive impedance having an impedance of $-jx$.



How to get the $-jx$?

A convenient way is to use a short-circuit or open-circuit **stub**, which is simply a transmission line terminated by a short- or open-circuit.

The idea is simple. If we look at the Smith chart again, the outer circle corresponds to $r = 0$, which means pure reactance or $\pm jx$.

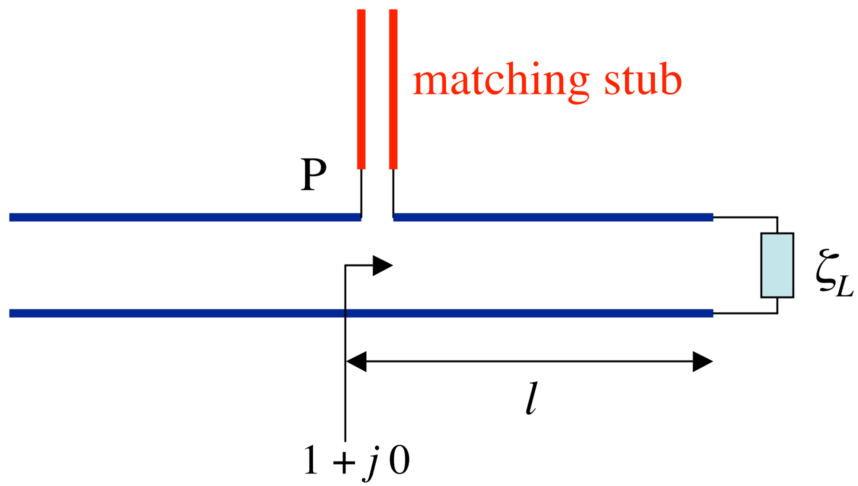


All we need to do is to find the appropriate length, starting with either a short- or open-circuit load, such that the required reactance is obtained. For example, if we need $-j1$, we just have a $\lambda/8$ line with open-circuit load, i.e., a $\lambda/8$ open-circuit stud.

Summary of Single Stub Matching Technique

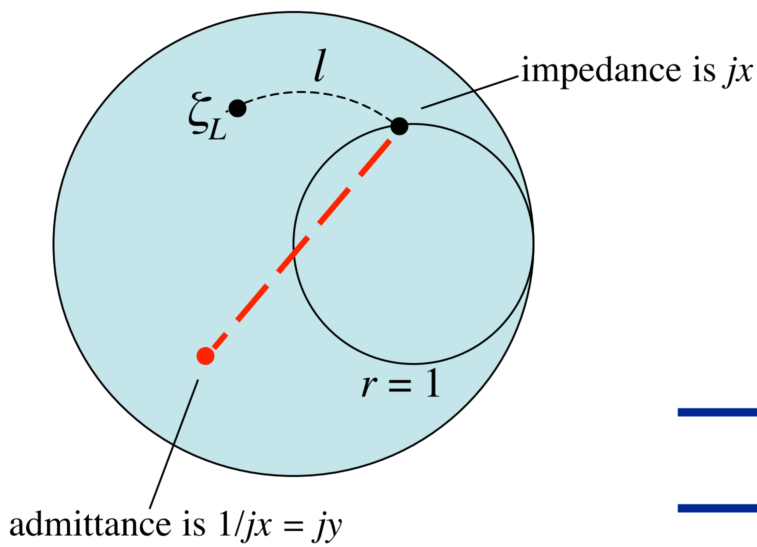
The above method, using one stub, is a common matching technique. The general procedure is:

1. Find the point at which $r = 1$. Cut it there.
2. Find the value of jx at that point.
3. Find the appropriate stub length to cancel out jx and connect the stud in series at that point.

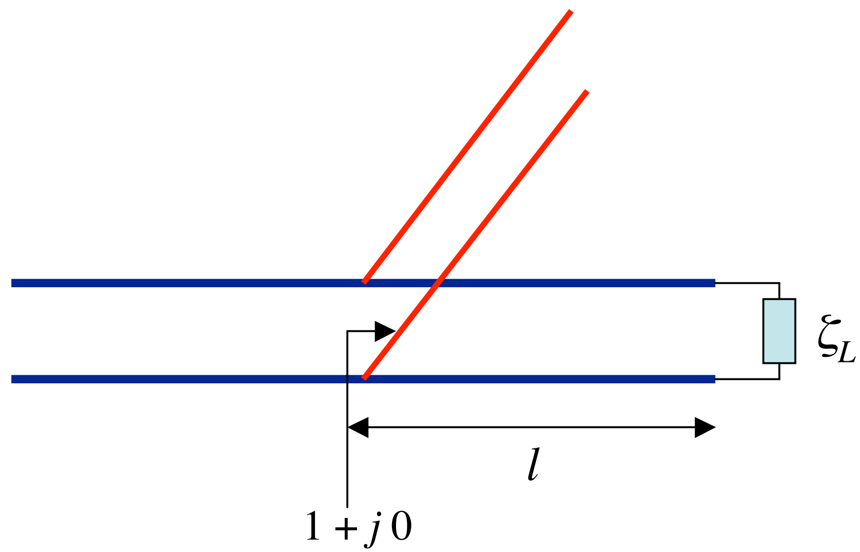


Obviously, the matching stub can also be connected in shunt (parallel), but the stub length may be different since we are now adding up admittance!

On the Smith chart, we should get the admittance of the required stub **by rotating the impedance by 180°.**



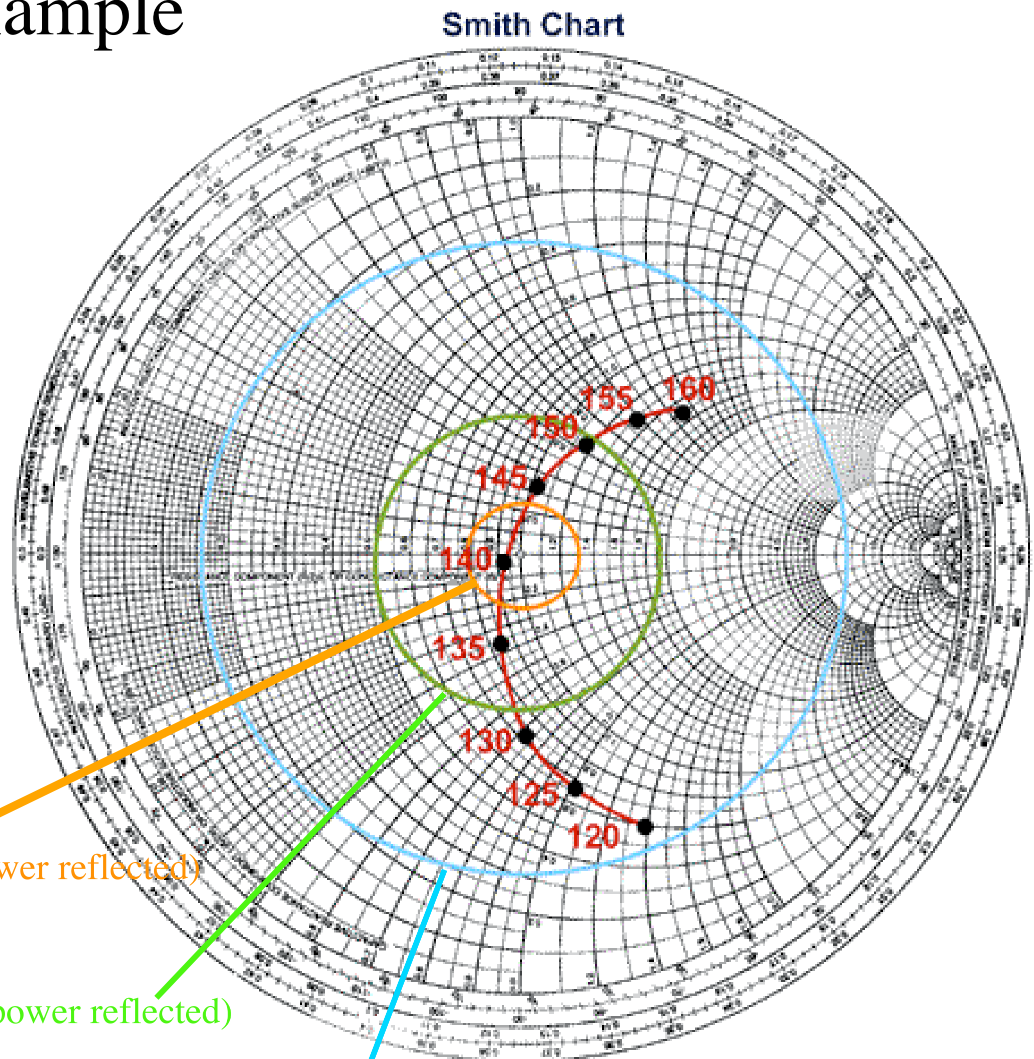
Thus, we can connect in shunt a stub of $-jy$ to cancel this jy.



Antenna Matching Example

Consider a 1-metre dipole antenna load which is used at a frequency not designed for. It is cheaper and easier to do a stub matching rather than to change the antenna structure. Suppose the frequency is to be changed from 120 MHz to 160 MHz.

Some initial experimental data of the normalized impedance at the dipole centre are plotted on the Smith chart, for 120, 125, 130, …, 160 MHz.



Matched at around 140 MHz (centre of chart). The circles represent constant Γ or VSWR and hence tell us about the bandwidth of this dipole:

VSWR=1.33 or $|\Gamma| = \sqrt{0.02} = 0.141$ (2% power reflected)
Frequency range is 137 MHz to 144 MHz

VSWR=1.93 or $|\Gamma| = \sqrt{0.1} = 0.316$ (10% power reflected)
Frequency range: 132 MHz to 151 MHz

Blue: VSWR=6 or $|\Gamma| = \sqrt{0.5} = 0.7071$ (50% power reflected)
Too bad! Significantly mismatched.

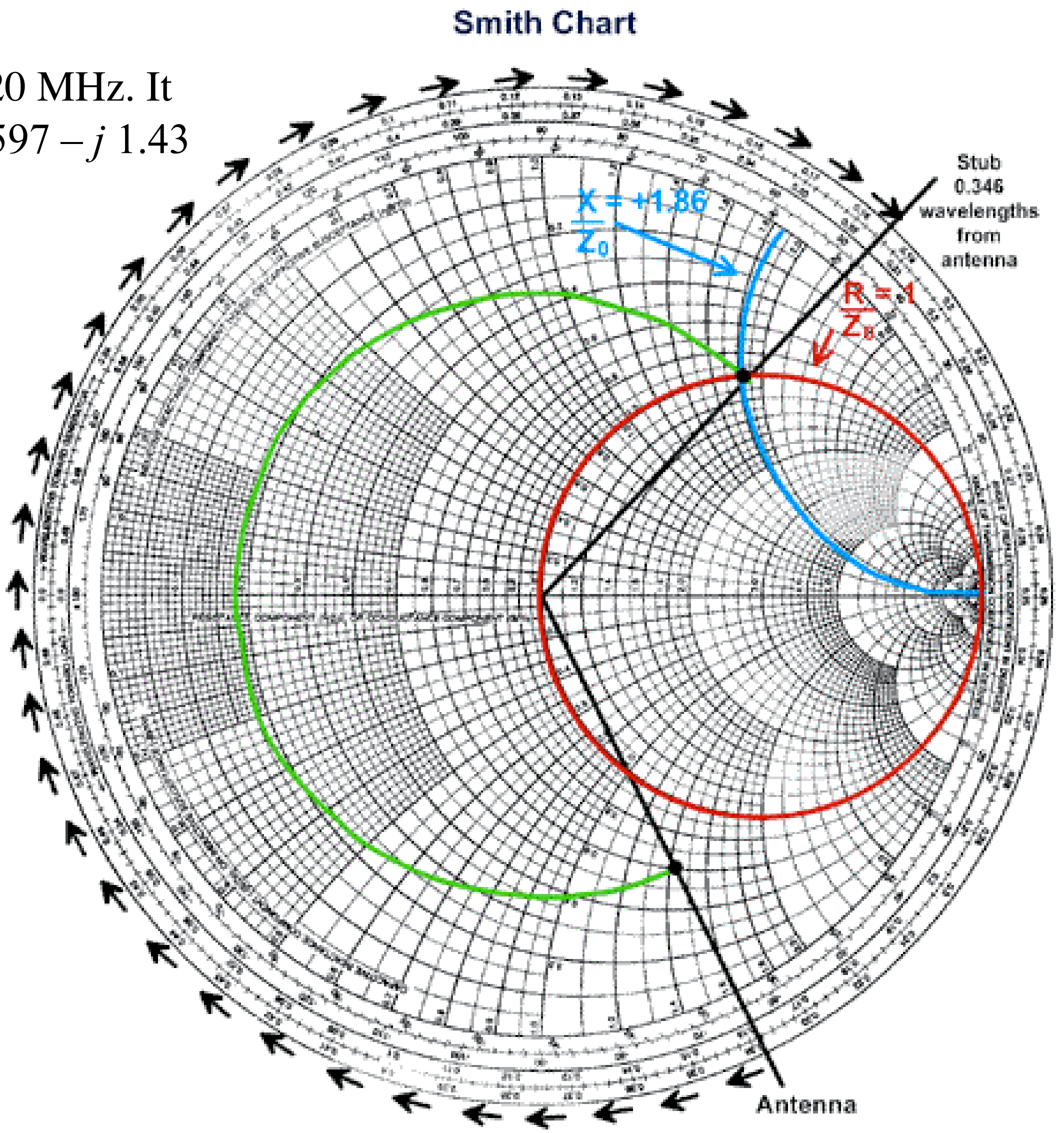
Matching Procedure

We begin with the impedance at 120 MHz. It is 44.8 – j 107 Ω, normalized to 0.597 – j 1.43 for a 75Ω coax line.

Rotate it clockwise (generator direction) until it reaches the circle corresponding to r = 1 (red circle). We see that

1. We need to insert a stub at 0.346λ from the antenna.
2. The normalized reactance at that point is j 1.86. So, we need to cancel this out.

Cancellation requires a short- or open-circuit stub of appropriate length.



Finally, to get the match stub length, we use the Smith chart again.

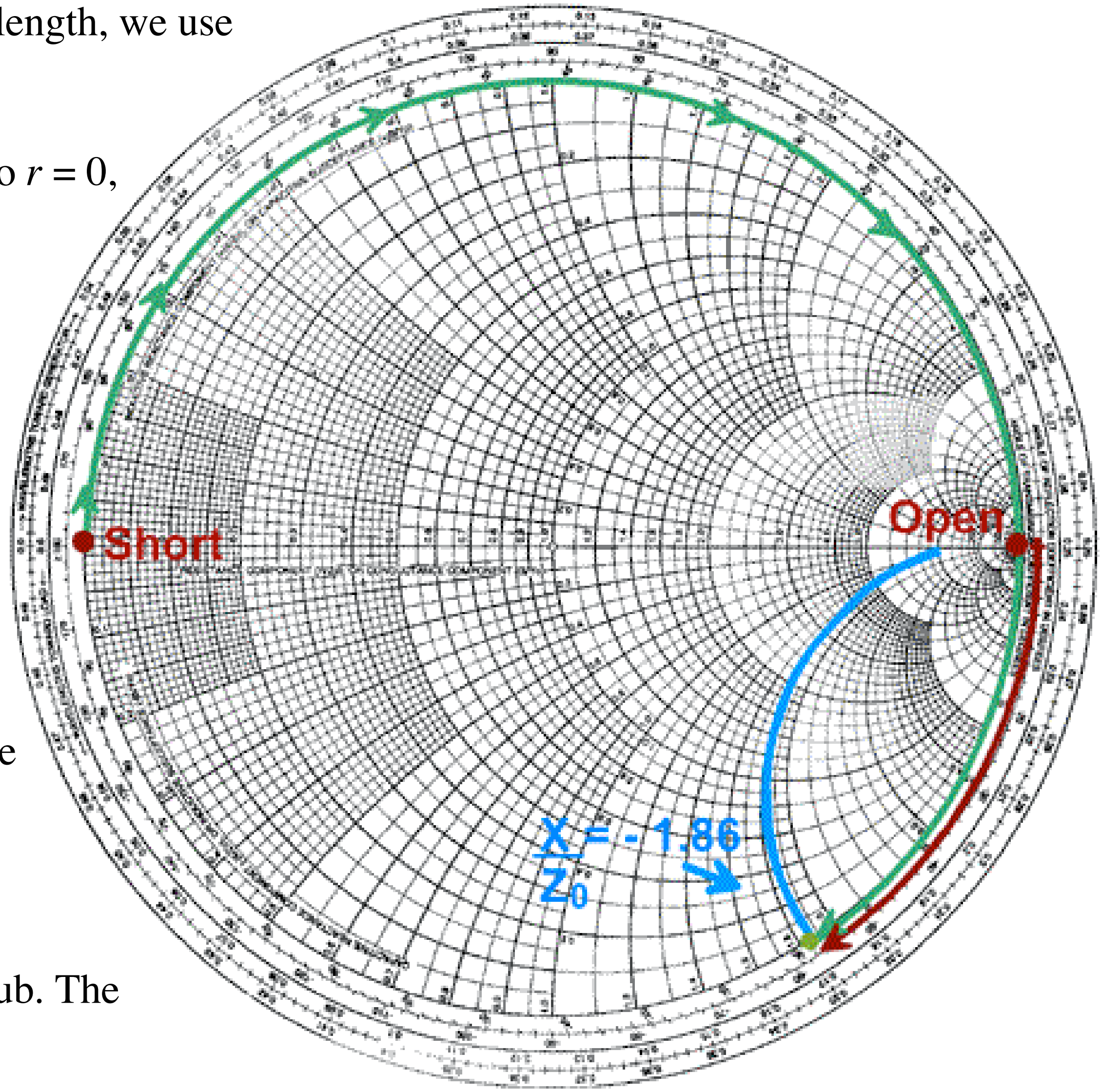
The outer circle corresponds to $r = 0$, i.e., pure reactance.

The leftmost point is SHORT, and the rightmost is OPEN.

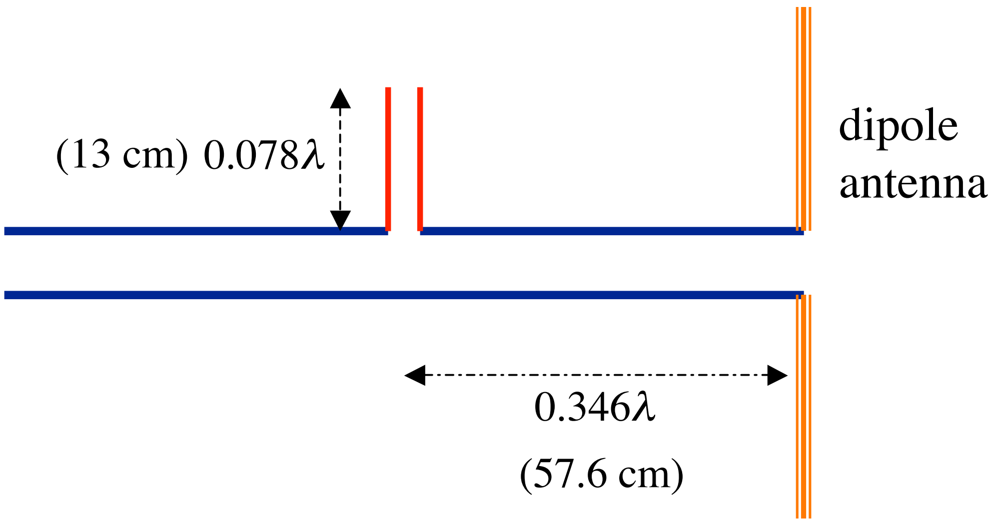
On the chart, we find the reactance circle of $x = -1.86$ and find the length required from either the OPEN or SHORT point.

Here, we need 0.328λ from the SHORT, or 0.078λ from the OPEN. **(Remember to go clockwise!)**

We choose the open-circuit stub. The required length is **0.078λ.**



Final Answer:



Suppose the wave velocity in the coax cable is 20 cm/ns (2/3 of light speed). Then, wavelength is 1.67 m.

So, $0.346\lambda = 0.5767$ m = **57.6 cm** and $0.078\lambda = 0.13$ m = **13 cm**.

Practical Points:

How long?
Make stub as short as possible for wider bandwidths, preferably less than $\lambda/2$. But if the stub is too short for precise cutting, a bit over $\lambda/2$ is acceptable. Remember that when you add $\lambda/2$, you get the same reactance value.

Series or shunt?
Physical construction usually dictates the choice. For balanced feeders like twin ribbon cables, series insertion is easy to make. But for coax, series is difficult.

Open or short stub?
If there is a choice, choose the one that makes the stub length shortest, preferably less than $\lambda/4$ if possible. For microstrips, open stubs are easier to make. For coax, short stubs are less radiating from ends.